SMOOTH LINEARIZATION OF COMMUTING CIRCLE DIFFEOMORPHISMS

BASSAM FAYAD AND KOSTANTIN KHANIN

Abstract. We show that a finite number of commuting diffeomorphisms with simultaneously Diophantine rotation numbers are smoothly conjugated to roations. This solves a problem raised by Moser in [5].

1. Introduction

In this paper, we show that if a finite number of commuting smooth circle diffeomorphisms have simultaneously Diophantine rotation numbers (arithmetic condition (1) below), then the diffeomorphisms are smoothly (and simultaneously) conjugated to rotations (see Theorem 1 below).

The problem of smooth linearization of commuting circle diffeomorphisms was raised by Moser in [5] in connection with the holonomy group of certain foliations with codimension 1. Using the rapidly convergent Nash-Moser iteration scheme he proved that if the rotation numbers of the diffeomorphisms satisfy a simultaneous Diophantine condition and if the diffeomorphisms are in some C^∞ neighborhood of the corresponding rotations (the neighborhood beign imposed by the constants appearing in the arithmetic condition, as usual in perturbative KAM theorems) then they are C^∞-linearizable, that is, C^∞-conjugated to rotations.

In terms of small divisors, the latter result presented a new and striking phenomenon: if d is the number of commuting diffeomorphisms, the rotation numbers of some or of all the diffeomorphisms may well be non-Diophantine, but still, the full $\mathbb{Z}^d$-action is smoothly linearizable due to the absence of simultaneous resonances. Further, Moser shows in his paper that this new phenomenon is a *genuine* one in the sense that the problem cannot be reduced to that of a single diffeomorphism with a Diophantine frequency. indeed he shows that there exist numbers $\theta_1, \ldots, \theta_d$ that are simultaneously Diophantine but such that for all linearly independent vectors $a, b \in \mathbb{Z}^{d+1}$, the ratios $(a_0+a_1\theta_1+\ldots+a_d\theta_d)/(b_0+b_1\theta_1+\ldots+b_d\theta_d)$ are Liouville numbers. In this case, the theory for individual circle maps, even the global theorem of Herman and Yoccoz, does not suffice to conclude.

According to Moser, the problem of linearizing commuting circle diffeomorphisms could be regarded as a model problem where KAM techniques can be applied to an overdtermined system (due to the commutation relations). This assertion could again be confirmed a quarter of a centuary later in a striking way by the recent work [2] where local rigidity of some higher rank abelian groups was established using a KAM scheme for an overdetermined system.

At the time Moser was writing his paper, the *global* theory of circle diffeomorphisms was already known for a while, of which a highlight result is that a diffeomoprhism with a Diophantine rotation number is smoothly linearizable (without a *local* condition of closeness to a rotation, see [6]). The proof of the first global smooth linearization theorem given by Herman, as well as all the

subsequent different proofs and generalizations, extensively used the Gauss algorithm of continued fractions that yields the best rational approximations for a real number.

As pointed out in Moser's paper, this is why the analogue global problem for a commuting family of diffeomorphisms with rotation numbers satisfying a simultaneous Diophantine condition seemed difficult to tackle, due precisely to the absence of an analogue of the one dimensional continuous fractions algorithm in the case of simultaneous approximations of several numbers (by rationals with the same denominator).

Moser asked *under which conditions on the rotation numbers of n smooth commuting circle diffeomorphisms can one assert the existence of a smooth invariant measure μ? In particular is the simultaneous Diophantine condition sufficient?* Here, we answer this question positively (Theorem 1, the existence of a smooth invariant measure being an equivalent statement to smooth conjugacy). On the other hand, it is not hard to see that the same arithmetic condition is optimal (even for the local problem) in the sense given by Remark 1.

Before we state our results and discuss the plan of the proofs, we give a brief summary of the linearization theory of single circle diffeomorphisms on which our proof relies.

We denote the circle by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We denote by $\mathrm{Diff}^r_+(\mathbb{T})$, $r \in [0, +\infty] \cup \{\omega\}$, the group of orientation preserving diffeomorphisms of the circle of class C^r or real analytic. We represent the lifts of these diffeomorphisms as elements of $D^r(\mathbb{T})$, the group of C^r-diffeomorphisms $\tilde{f}$ of the real line such that $f - \mathrm{Id}_{\mathbb{R}}$ is $\mathbb{Z}$-periodic.

Following Poincaré, one can define the rotation number of a circle homeomorphism f as the uniform limit $\rho_f = \lim(\tilde{f}^j(x) - x)/j \mathrm{mod}[1]$, where $\tilde{f}^j$ $(j \in \mathbb{Z})$ denote the iterates of a lift of f. A rotation map of the circle with angle θ, that we denote by $R_\theta : x \mapsto x + \theta$, has clearly a rotation number equal to θ. Poincaré raised the problem of comparing the dynamics of a homeomorphism of the circle with rotation number θ to the simple rotation R_θ.

A classical result of Denjoy (1932) asserts that if $\rho_f = \theta$ is irrational (not in $\mathbb{Q}$) and if f is of class C^1 and if Df has bounded variations then f is topologically conjugated to R_θ, i.e. there exists a circle homeomorphism h such that $h \circ f \circ h^{-1} = R_\theta$.

Considering the linearized version of the conjugation equation $H(x + \theta) - H(x) = F(x)$ where H and F are real $\mathbb{Z}$-periodic functions defined on $\mathbb{R}$ and where F is assumed to have zero mean, it is easy to see (with Fourier analysis, due to the existence of the *small divisors* $|1 - e^{i2\pi n\theta}|$) that the existence of a smooth solution H, is guaranteed for all functions F with zero mean if and only if θ satisfies a Diophantine condition, i.e. if there exist $C > 0$ and $\tau > 0$ such that for any $k \in \mathbb{Z}$, $\|k\theta\| \geq C|k|^{-\tau}$. Nonetheless, when F is in some finite class of differentiability and the linearized equation has a solution, this solution in general is of lower regularity than F. This is the so-called *loss of regularity* phenomenon.

The first result asserting regularity of the conjugation of a circle diffeomorphism to a rotation was obtained by Arnol'd in the real analytic case: if the rotation number of a real analytic diffeomorphism is Diophantine and if the diffeomorphism is sufficiently close to a rotation, then the conjugation is analytic. The general idea, that is due to Kolmogorov, is to use a quadratic Newton approximation method to show that if we start with a map sufficiently close to the rotation it is possible to compose successive conjugations and get closer and

closer to the rotation while the successive conjugating maps tend rapidly to the Identity. The Diophantine condition is used to insure that the loss of differentiability in the linearized equation is fixed, which allows to compensate this loss at each step of the algorithm due to its quadratic convergence. Applying the same Newton scheme in the C^∞ setting is essentially due to Moser.

At the same time, Arnol'd also gave examples of real analytic diffeomorphisms with irrational rotation numbers for which the conjugating maps are not even absolutely continuous, thus showing that the small divisors effect was inherent to the regularity problem of the conjugation. Herman also showed that there exist "pathological" examples for any non-Diophantine irrational (i.e. Liouville) rotation number (see [4, chap. XI], see also [3]).

A crucial conjecture was that, to the contrary, the hypothesis of closeness to rotations should not be necessary for smooth linearization, that is, any smooth diffeomorphism of the circle with a Diophantine rotation number must be smoothly conjugated to a rotation. This *global* statement was finally proved by Herman in [4] for almost every rotation number, and later on by Yoccoz in [6] for all Diophantine numbers.

To solve the global conjecture, Herman, and later on Yoccoz, developped a powerful machinery giving sharp estimates on derivatives growth for the iterates of circle diffeomorphisms, the essential criterion for the C^r regularity of the conjugation of a C^k diffeomorphism f, $k \geq r \geq 1$, being the fact that the family of iterates (f^n) should be bounded in the C^r topology. The Herman-Yoccoz estimates on the growth of derivatives of the iterates of f will be crucial for us in all the paper.

2. Results

For $\theta \in \mathbb{T}$ and $r \in [1, +\infty] \cup \{\omega\}$, we denote by $\mathcal{D}^r_\theta$ the subset of $\mathrm{Diff}^\infty_+(\mathbb{T})$ of diffeomorphisms having rotation number θ.

Let $d \in \mathbb{N}$, $d \geq 2$, and assume that $(\theta_1, \dots, \theta_d) \in \mathbb{T}^d$ are such that there exist $\nu > 0$ and $C > 0$ such that for each $k \in \mathbb{Z}^*$,

$$\max(\|k\theta_1\|, \dots, \|k\theta_d\|) \geq C|k|^{-\nu}. \tag{1}$$

Finally, we say that a family of circle diffeomorphisms $(f_1, \dots, f_d)$ is *commuting* if $f_i \circ f_j = f_j \circ f_i$ for all $1 \leq i \leq j \leq p$. Note that if h is a homeomorphism of the circle such that $h \circ f_1 \circ h^{-1} = R_{\theta_1}$, then for every $j \leq p$ we have that $h \circ f_j \circ h^{-1}$ commutes with R_{θ_1}, from which it is easy to see that $h \circ f_j \circ h^{-1} = R_{\theta_j}$. Hence, for $r \geq 2$, Denjoy theory gives a homeomorphism that conjugates every f_j to the corresponding rotation. Here, we prove the following.

Theorem 1. *Assume that $\theta_1, \dots, \theta_d$ satisfy (1) and let $f_i \in \mathcal{D}^\infty_{\theta_i}$, $i = 1, \dots, p$. If $(f_1, \dots, f_d)$ is commuting then, there exists $h \in \mathrm{Diff}^\infty_+(\mathbb{T})$, such that for each $1 \leq i \leq p$, $h \circ f_i \circ h^{-1} = R_{\theta_i}$.*

Remark 1. Using Liouvillean constructions (constructions by successive conjugations) we see that the above sufficient arithmetic condition is also necessary to guarantee some regularity on the conujugating homeomorphism h (essentially unique, up to translation). There is indeed a sharp dichotomy with the above statement in case the arithmetic condition is not satisfied (see for example [4, chap. XI] and [3] where the same techniques producing a single diffeomorphism readily apply to our context): *Assume that $\theta_1, \dots, \theta_d$ do not satisfy (1), then there exist $f_i \in \mathcal{D}^\infty_{\theta_i}$, $i = 1, \dots, p$ such that $(f_1, \dots, f_d)$ is commuting and such*

that the conjugating homeomorphism of the maps f_i to the rotations R_{θ_i} is not absolutely continuous.

As a corollary of Theorem 1 and of the local theorem (on commuting diffeomorphisms) of Moser in the real anlytic category [5] we have by the same techniques as in [4, chap. XI. 6]:

Corollary 1. *Assume that $\theta_1, \ldots, \theta_d$ satisfy (1) and let $f_i \in \mathcal{D}^{\omega}_{\theta_i}$, $i = 1, \ldots, p$. If $(f_1, \ldots, f_d)$ is commuting then, there exists $h \in \mathrm{Diff}^{\omega}_{+}(\mathbb{T})$, such that for each $1 \leq i \leq p$, $h \circ f_i \circ h^{-1} = R_{\theta_i}$.*

If $(\theta_1, \ldots, \theta_d) \in \mathbb{T}^p$ are such that there exist $a \in (0, 1)$ and infinitely many $k \in \mathbb{N}$ such that

$$\max(\|k\theta_1\|, \ldots, \|k\theta_d\|) \leq a^k$$

then it is possible to obtain with a construction by successive conjugations a commuting family $(f_1, \ldots, f_d) \in \mathcal{D}^{\omega}_{\theta_1} \times \ldots \times \mathcal{D}^{\omega}_{\theta_d}$ such that the conjugating homeomorphism of the maps f_i to the rotations R_{θ_i} is not absolutely continuous.

It is a delicate problem however to find the optimal arithmetic condition under which any commuting family of real analytic diffeomorphisms will be linearizable in the real analytic category. For a single real analytic diffeomorphism, the optimal condition was obtained by Yoccoz in [7].

3. Plan of the proof of Theorem 1

As in the global theory of circle diffeomorphisms, we will start by proving the C^1 regularity of the conjugation and then we will derive from it by Hadamard convexity inequalities and bootstrap techniques the C^∞ regularity. In each of these two moments of the proof the commutativity of the diffeomorphisms in question will be used differently.

The first step in the proof is a simple arithmetic observation for which we need the following definition: given an angle θ we say that a sequence of successive denominators of θ, $q_l, q_{l+1}, \ldots, q_n$, is a *Diophantine string of exponent* $\tau > 0$ if for all $s \in [l, n-1]$, $q_{s+1} \leq q_s^{\tau}$. The observation is that if we consider a sufficiently large number of angles $\theta_1, \ldots, \theta_p$ such that each d-upple satisfies (1) then we can find Diophantine strings of the same exponent τ (function of ν and d) for different θ_j's, such that these strings overlap (with a margin that can be made as large as the number of angles considered is large). In other words, one can follow successive denominators along a Diophantine string i until its end, say at some q_{j_i, n_i}, where it is possible to switch to the next string $i+1$ starting from a denominator $q_{j_{i+1}, l_{i+1}}$ that is well smaller than q_{j_i, n_i} ($q_{j_{i+1}, l_{i+1}} \leq q_{j_i, n_i}^{\xi}$, ξ as small as desired as the number p increases. The next elementary but crucial observation is that given $f_1, \ldots, f_d$ with rotation numbers $\theta_1, \ldots, \theta_d$ satisfying (1), it is possible, by considering compositions of these diffeomorphisms to obtain as much diffeomorphisms as desired with rotation numbers in such a way that any d-upple satisfies (1). Sections 4 and 5 deal with these results on the alternated configuration of Diophantine strings.

With this configuration in hand the proof of C^1-conjugacy goes as follows: first, to alleviate the notations we will consider only the case $d = 2$ (the proof for $d \geq 3$ is exactly the same) and assume that the Diophantine strings of $\theta = \rho_{f_1}$ and $\beta = \rho_{f_2}$ are themselves in an alternated configuration (Conditions (4)–(6)) since this also does not make any difference with the proof in the general case.

If we denote by m_n and M_n the minimum and the maximum on the circle of $|x - f^{q_n}(x)|$ (where q_n denotes the denominators of the convergents of θ, and with similar notations $\tilde{q}_n$, $\tilde{m}_n$, and $\tilde{M}_n$ for β and g), then a criterion for C^1-conjugacy of f to a rotation is that m_n/M_n be bounded. It is known that $m_n \leq \theta_n \leq M_n$ where $\theta_n = |q_n\theta - p_n|$ and the goal is to show that eventually both m_n and M_n become comparable to θ_n up to a multiplicative constant. In [6] a crucial recurrence relation between these quantities at the steps n and $n+1$ is exhibited that allows to show, if a Diophantine condition holds on θ, that the quantities m_n and M_n end up having the same order. The latter recurrence relation is obtained as a result of the analysis of the growth of the Schwartzian derivatives of the iterates of f.

Here we will rely on the same recurrence relation but use it only along the Diophantine strings and try to propagate the improvement of estimations when we switch strings using the commutation of f and g. Actually this will work efficiently once we get started, namely once M_s for q_s in some Diophantine string for θ is less than $\theta_s^{1-\sigma}$ for some fixed $\sigma > 0$ that depends on τ (it is possible to take $\sigma = 1/(2\tau^2)$). This can be interpreted as a "local" result that yields C^1 conjugation for diffeomorphisms that are close to rotations (see Proposition 5).

The existence of very long Diophantine strings (which corresponds to one of the angles being super-Liouvillean) presents the simplest case illustrating how the local situation can indeed be reached using only one string (see Section 6.3).

In general however, before reaching the local situation, switching from a string to a consecutive one may in fact lead to a worsening of the estimates or even to their complete loss (see the first equation in the proof of Lemma 3), so that a different strategy must be adopted. Keeping in mind that the objective is to see that $u_s \to 1$ where u_s is such that $M_s = \theta_s^{u_s}$ (with $\tilde{M}_s$, $\tilde{u}_s$, and β_s for β and g), the idea is to use each angle alone to study "the dynamics" of u_s: after we measure the gain in the exponent u when we pass through a Diophantine string, we jump to the beginning of the successive string of the same angle. In this operation we can readily limit the loss in the exponent u in function of the size of the jump (that in turn is less than the size of the overlapping Diophantine string of the other angle). Repeating these two steps inductively, we get a dynamics on the exponent u_i measured at the exit of the i^{th} Diophantine string (of the same angle, see Lemma 4). Doing so for each angle we see that at least for one of them, namely the one with the overall longest Diophantine strings (in the sense given by (18) or (19)), the sequence u_i (or $\tilde{u}_i$) eventually becomes larger than $1 - \sigma$.

The idea for proving higher regularity is to use convexity arguments as in [4, 6] to bound the derivatives of the iterates of f and g. To this difference that we will only seek to bound these derivatives for iterates f^u and g^v at *Diophantine times* u and v that are (respectively) linear combinations of multiples of denominators q_s and $\tilde{q}_s$ that belong to Diophantine strings (each q_s is as usual multiplied by at most q_{s+1}/q_s). Due to the overlapping of strings, this will be sufficient for proving regularity of the conjugation (see Section 7.1).

Given a denominator q_s in a Diophantine string, the fact that the ratio q_{s+1}/q_s is bounded by a fixed power of q_s is naturally crucial in the control of the dervia- tives of the diffeomorphisms $f^{aq_s}, a \leq q_{s+1}/q_s$. Nonetheless, in the Herman-Yoccoz theory for circle diffeomorphisms with Diophantine rotation number, the control of the derivatives of f^{q_s} itself are obtained using the whole Diophantine condition on the diffeomorphism's rotation number (see the computations in [6,

section 8]). Still, we can see through the calculations of [6, section 8] (see Section 7.2 below), that the existence of a sufficiently long sequence of Diophantine string before and up to some denominator q_s, combined with the existence of a C^1-conjugacy to a rotation, allows to give a bound on the derivatives of f^{q_s} that will be enough for our purpouse.

Thus, in addition to the alternation of Diophantine strings used for C^1 we must make sure that there is enough Diophantine "margin" before q_{l_i} and this is done (in Proposition 2) through the use of even more numbers θ_i, which amounts to considering in the proof more diffeomorphisms of the form $f^i \circ g$. In a sense we use more and more relations in the commuting group of diffeomorphisms as we want to improve the regularity of the conjugation.

The rest of the proof of higher regularity is inspired by the bootstrap calculations of [6].

Nowhere in our proof of Theorem 1, neither in the proof of the existence of C^1-conjugation nor in that of its higher regularity, did we try to optimize on our *use of derivatives* of the diffeomorphism f, that is assumed to be of class C^∞. For instance, the problem of finding the optimal regularity required on the diffeomorphisms that would guarantee C^1-conjugation under a given simultaneous Diophantine condition is an interesting problem that is not addressed in this paper.

4. Preliminary : Diophantine strings

We recall that for every irrational number θ we can uniquely define an increasing sequence of integers q_n such that $q_1 = 1$, and

$$\|k\theta\| > \|q_n\theta\|, \quad \forall k < q_{n+1}, k \neq q_n.$$

This sequence is called the sequence of denominators of the best rational approximations, or convergents, of α.

Let $p \in \mathbb{N}$, and $\theta_1, \dots, \theta_p$ be irrational numbers. For $1 \leq j \leq p$, we denote by $(q_{j,n})$ the sequence of denominators of the convergents of θ_j. For $\tau > 0$, we define

$$\mathcal{A}_\tau(\theta_j) = \{s \in \mathbb{N} \ / \ q_{j,s+1} \leq q_{j,s}^\tau\}.$$

A Diophantine string (with exponent τ) for a number θ_i is then a sequence $l, l+1, \dots, n-1 \in \mathcal{A}_\tau(\theta_i)$.

We will prove in this section the main arithmetical result related with the simultaneous Diophantine property (1) that we will use to prove Theorem 1.

Proposition 1. *Let $\nu > 0$, $K > 0$ and $d \in \mathbb{N}$, $d \geq 2$. There exists $p \in \mathbb{N}$ such that: if $\theta_1, \dots, \theta_p$ are numbers for which there exists $C > 0$ such that each $d-$upple (of disjoint numbers) $(\theta_{i_1}, \dots, \theta_{i_d})$ satisfies (1); if $U > 0$ is sufficiently large and if $U \leq V \leq U^K$; then there exists $k \in \{1, \dots, p\}$, with a Diophantine string $l, \dots, n-1 \in \mathcal{A}_{\tau_{d-1}}(\theta_k)$ with*

$$q_{k,l} \leq U \leq V \leq q_{k,n},$$

where (τ_s) is the sequence defined by $\tau_0 = \nu$ and $\tau_s = 2\tau_{s-1} + 3$, for $s \geq 1$.

Definition 1. *For $\tau > 0$, $C > 0$, $d \in \mathbb{N}^*$, and an interval $I \subset \mathbb{R}$ we define*

$$D_{d,\tau,C}(I) = \{(\theta_1, \dots, \theta_d) \in \mathbb{R}^d \ / \ \sup_{1 \leq i \leq d} \|k\theta_i\| \geq Ck^{-\tau}, \ \forall k \in I\}.$$

For $C = 1$, we use the simplified notation $D_{d,\tau}(I) := D_{d,\tau,1}(I)$.

We will need the following elementary but crucial arguing.

Lemma 1. *Let $\nu > 0, C > 0, d \in \mathbb{N}, d \geq 2$. Define $\epsilon = 1/(2\nu+2)$. There exists U_0 such that if $V \geq U \geq U_0$, and if $\theta_1, \dots, \theta_d$ are numbers such that*

$$(\theta_1, \dots, \theta_d) \in D_{d,\nu,C}([U,V]),$$

then if an integer $s \in [U,V]$ satisfies $\|s\theta_d\| \leq s^{-(2\nu+3)}$, we have that

$$(\theta_2, \dots, \theta_{d-1}) \in D_{d-1,2\nu+3}([s,e])$$

with $e = \min(V, \|s\theta_d\|^{-\epsilon})$.

Proof. If $k \in [s,e]$ satisfies

$$\sup_{i \leq d-1} \|k\theta_i\| \leq k^{-(2\nu+3)},$$

then

$$\sup_{i \leq d} \|ks\theta_i\| \leq (ks)^{-(\nu+\frac{1}{2})},$$

which violates $(\theta_1, \dots, \theta_d) \in D_{d,\nu,C}([U,V])$, if s is sufficiently large. $\square$

Because $\eta = (2\nu+3)/(2\nu+2) > 1$, Lemma 1 has the following immediate consequence.

Corollary 2. *Let $\nu > 0$, $K > 0$ and $d \in \mathbb{N}$, $d \geq 2$. There exists $N \in \mathbb{N}$ such that: for each $C > 0$, there exists $U_0 > 0$, such that if $U \geq U_0$ and $U \leq V \leq U^K$, and if $p \geq N + d - 1$ and $\theta_1, \dots, \theta_p$ are numbers such that for each $d-$upple (of disjoint indices) $i_1, \dots, i_d$, $(\theta_{i_1}, \dots, \theta_{i_d}) \in D_{d,\nu,C}([U,V])$, then there exist $j_1, \dots, j_N \leq p$ such that any $(d-1)$-upple (of disjoint numbers), $i_1, \dots, i_{d-1} \in \{1, \dots, p\} - \{j_1, \dots, j_N\}$, satisfies $(\theta_{i_1}, \dots, \theta_{i_{d-1}}) \in D_{d-1,2\nu+3}([U,V])$.*

Proof. We can in fact take $N = [\ln K/\ln \eta] + 2$. Let $p \geq N + d - 1$ and let $k_1 \in \mathbb{N}$, $k_1 \geq U$, be the smallest integer (if it exists) such that $\|k_1\theta_i\| \leq {k_1}^{-(2\nu+3)}$ for some $i \in \{1, \dots, p\}$. Denote by θ_{j_1} the corresponding angle. Then, define $k_2 \geq \|k_1\theta_i\|^{-(2\nu+3)}$, to be the smallest integer (if it exists) such that $\|k_2\theta_i\| \leq k_1{}^{-(2\nu+3)}$ for some $i \in \{1, \dots, p\} - \{j_1\}$ and denote by θ_{j_2} the corresponding angle. Continuing this way, we construct a sequence $j_1, \dots, j_N$, and observe that $k_N \geq k_1^{\eta^N} > V$. On the other hand, Lemma 1 implies that any $(d-1)$-upple (of disjoint numbers), $i_1, \dots, i_{d-1} \in \{1, \dots, p\} - \{j_1, \dots, j_N\}$, satisfies $(\theta_{i_1}, \dots, \theta_{i_{d-1}}) \in D_{d-1,2\nu+3}([k_s, \|k_s\theta_{j_s}\|^{-(2\nu+3)}])$. But, by definition of $K_1, \dots, k_N$, for every $i \in \{1, \dots, p\} - \{j_1, \dots, j_N\}$, and for every $k \in [U, k_1) \cup (\|k_1\theta_{j_1}\|^{-(2\nu+3)}, k_2) \cup \dots \cup (\|k_{N-1}\theta_{j_{N-1}}\|^{-(2\nu+3)}, k_N)$, we have $k\theta_i \geq \|k\theta_i\|^{-(2\nu+3)}$. Thus, $(\theta_{i_1}, \dots, \theta_{i_{d-1}}) \in D_{d-1,2\nu+3}([U,V])$. $\square$

Proof of Proposition 1. If p and U are sufficiently large, applying Corollary 2 $d-1$ times (with $U^{1/(2\tau_{d-1})}$ instead of U) we get that there exists $k \in \{1, \dots, p\}$ such that $\theta_k \in D_{1,\tau_{d-1}}([U^{1/(2\tau_{d-1})}, V])$. We claim that θ_k satisfies the properties required in Proposition 1. Indeed, it is sufficient to prove that θ_k must have a denominator $q_{k,l} \in [U^{1/(2\tau_{d-1})}, U]$. But if this is not so, there is some $q_{k,l} \leq U^{1/(2\tau_{d-1})}$ such that $q_{k,l+1} \geq U$, but then $m = q_{k,l}U^{1/(2\tau_{d-1})} \leq U$ satisfies $\|m\theta_k\| \leq m^{-\tau_{d-1}}$, in contradiction with $\theta_k \in D_{1,\tau_{d-1}}([U^{1/(2\tau_{d-1})}, V])$. $\square$

5. Alternated configuration of denominators

Definition 2. *We say that $\theta_1, \ldots, \theta_p$ are in an alternated configuartion if there exist $\tau > 1$, and two increasing sequences of integers, l_i and n_i such that for each i there exists $j_i \in \{1, \ldots, p\}$ with*

$$l_i, l_i + 1, l_i + 2, \ldots, n_i - 1 \in \mathcal{A}_\tau(\theta_{j_i}), \tag{2}$$

and

$$q_{j_i,l_i}^{\tau^2} \leq q_{j_i,n_i}^{\frac{1}{\tau^2}} \leq q_{j_{i+1},l_{i+1}} \leq q_{j_i,n_i}^{\frac{1}{\tau}}. \tag{3}$$

From Proposition 1 it is straightforward to derive the following

Proposition 2. *Let $\nu > 0$, $\xi > 0$, and $d \in \mathbb{N}$, $d \geq 2$. Let $\tau := \tau_{d-1}$. There exists $p \in \mathbb{N}$ such that if $\theta_1, \ldots, \theta_p$ are numbers for which there exists $C > 0$ such that each $d-$upple (of disjoint numbers) $(\theta_{i_1}, \ldots, \theta_{i_d})$ satisfies (1) then $\theta_1, \ldots, \theta_p$ are in an alternated configuration (with exponent τ) with in addition that for each i there exists l'_i such that $q_{j_i,l'_i} \leq q_{j_i,l_i}^{\xi}$ and such that $l'_i, l'_i + 1, \ldots, l_i - 1 \in \mathcal{A}_\tau(\theta_{j_i})$.*

In our proof of Theorem 1, we will show that if $f_1, \ldots, f_p$ are smooth commuting diffeomorphisms with rotation numbers $\theta_1, \ldots, \theta_p$ that are in an alternated configuration, then the diffeomorphisms are C^1-conjugated to rotations. The additional condition, i.e. the existence of long Diophantine strings before q_{l_i} is then used to proof the higher regularity of the conjugacy, the higher the regularity required, the longer these Diophantine strings should be ($\xi \to 0$).

To adapt Proposition 2 to a family of d commuting diffeomorphisms, we use the following somehow artificial trick[1]: consider $\theta_1, \ldots, \theta_d$ satisfying (1) and define for $s \in \mathbb{N}$

$$\tilde{\theta}_s = \theta_1 + s\theta_2 + \ldots + s^{d-1}\theta_d.$$

Observe that for any $p \geq d$, there exists $C > 0$ such that any disjoint indices $i_1, \ldots, i_d \leq p$, we have that $(\tilde{\theta}_{i_1}, \ldots, \tilde{\theta}_{i_d})$ satisfies (1). Proposition 2 can now be applied to $\tilde{\theta}_1, \ldots, \tilde{\theta}_p$. On the other hand, given $f_1, \ldots, f_d$ as in Theorem 1, then the diffeomorphism $\tilde{f}_s = f_1 \circ f_2^s \circ f_3^{s^2} \circ \ldots \circ f_d^{s^{d-1}}$ has rotation number $\tilde{\theta}_s$.

Since it does not alter the proof but only alleviates the notations we will assume for the sequel that $d = 2$ and that θ and β are already in an alternated configuration, that is, there exist $\tau > 1$, and two increasing sequences of integers, l_i and n_i such that

$$l_{2i}, \ldots, n_{2i} - 1 \in \mathcal{A}_\tau(\theta) \tag{4}$$

$$l_{2i+1}, \ldots, n_{2i+1} - 1 \in \mathcal{A}_\tau(\beta) \tag{5}$$

and

$$q_{l_{2i}}^{\tau^2} \leq q_{n_{2i}}^{\frac{1}{\tau^2}} \leq \tilde{q}_{l_{2i+1}} \leq q_{n_{2i}}^{\frac{1}{\tau}}, \quad \tilde{q}_{l_{2i+1}}^{\tau^2} \leq \tilde{q}_{n_{2i+1}}^{\frac{1}{\tau^2}} \leq q_{l_{2i+2}} \leq \tilde{q}_{n_{2i+1}}^{\frac{1}{\tau}}. \tag{6}$$

where (q_n) and $(\tilde{q}_n)$ denote respectively the sequences of denominators of the convergents of θ and β.

6. Proof of C^1-conjugation

Given θ and β satisfying (4)–(6) and two commuting diffeomorphisms $f \in \mathcal{D}_\theta$, $g \in \mathcal{D}_\beta$ we will show in this section that f and g are C^1-conjugated to the rotations R_θ and R_β.

[1]We may attribute, as we did in the introduction, the usefulness of this trick to the fact that it exploits the raltions in the group, isomorphic to $\mathbb{Z}^d$, of commuting diffeomorphisms.

6.1. Let

$$\begin{aligned} \theta_n &= |q_n\theta - p_n|, & \beta_n &= |\tilde{q}_n\beta - \tilde{p}_n| \\ M_n &= \sup d(f^{q_n}(x), x), & \tilde{M}_n &= \sup d(g^{\tilde{q}_n}(x), x) \\ m_n &= \inf d(f^{q_n}(x), x), & \tilde{m}_n &= \inf d(g^{\tilde{q}_n}(x), x) \\ U_n &= \frac{M_n}{m_n}, & \tilde{U}_n &= \frac{\tilde{M}_n}{\tilde{m}_n}. \end{aligned}$$

Recall that

$$1/(q_{n+1}+q_n) \leq \theta_n \leq 1/q_{n+1}, \quad 1/(\tilde{q}_{n+1}+\tilde{q}_n) \leq \beta_n \leq 1/\tilde{q}_{n+1}. \tag{7}$$

Recall also that since $\int_{\mathbb{T}} |f^{q_n} - \mathrm{id}| d\mu = \theta_n$, (where μ is the unique probability measure invariant by f) then

$$m_n \leq \theta_n \leq M_n.$$

Herman proved that a diffeomorphism is C^r conjugated to a rotation if and only if its iterates form a bounded sequence in the C^r-topology (see [4, Chap. IV]). Based on the latter observation, the following criterion for C^1 conjugacy was used in [4] and in [6, section 7.6]:

Proposition 3. *If there exists $C > 0$ such that $\limsup U_n \leq C$, then f is C^1-conjugated to R_θ (actually $\liminf U_n \leq C$ is enough).*

Our proof of C^1-conjugacy in Theorem 1 relies on the following central estimate of [6]

Proposition 4. *For any $f \in \mathcal{D}_\theta$, for any $K \in \mathbb{N}$, there exists $C = C(f, K)$ such that*

$$M_n \leq M_{n-1} \frac{(\theta_n/\theta_{n-1}) + CM_{n-1}^K}{1 - CM_{n-1}^{1/2}} \tag{8}$$

$$m_n \geq m_{n-1} \frac{(\theta_n/\theta_{n-1}) - CM_{n-1}^K}{1 + CM_{n-1}^{1/2}}. \tag{9}$$

6.2. The goal of this section is to prove the following "local" result:

Proposition 5. *Let $\sigma = 1/(2\tau^2)$. There exists $i_0 \in \mathbb{N}$ such that if for some even (odd) integer $i \geq i_0$, we have*

$$M_{n_i - 1} \leq \frac{1}{q_{n_i}^{1-\sigma}}$$

(with $\tilde{M}_{n_i-1}$ and $\tilde{q}_{n_i}$ instead of M_{n_i-1} and q_{n_i} if i is odd) then U_n and $\tilde{U}_n$ are bounded.

Remark 2. This can be viewed as a local result on C^1-conjugation, since it states that if M_{n_i-1} for i sufficiently large is not too far from what it should be if f were C^1-conjugated to the rotations, then f and g must indeed be C^1-conjugated to the rotations.

Proof of Proposition 5. We will assume that i is even, the other case being similar. Due to the commutation of f and g we have

Lemma 2. *Let* $L_i = [\beta_{l_{i+1}-1}/\theta_{n_i-1}]$, *then*

$$\begin{aligned}\tilde{M}_{l_{i+1}-1} &\leq (1+L_i)M_{n_i-1} & (10)\\ \tilde{m}_{l_{i+1}-1} &\geq L_i m_{n_i-1} & (11)\\ \tilde{U}_{l_{i+1}-1} &\leq (1+\frac{1}{L_i})U_{n_i-1}. & (12)\end{aligned}$$

Proof. If we assume that l_{i+1} and n_i (the other case being similar) we observe that for any $x \in \mathbb{T}$,

$$\bigcup_{k=0}^{L_i-1} R_\theta^{kq_{n_i-1}}([x, R_\theta^{q_{n_i-1}}(x)]) \subset [x, R_\beta^{\tilde{q}_{l_{i+1}-1}}(x)] \subset \bigcup_{k=0}^{L_i} R_\theta^{kq_{n_i-1}}([x, R_\theta^{q_{n_i-1}}(x)]). \quad (13)$$

Since f and g commute there exists a continuous homeomorphism h that conjugates f to R_θ and g to R_β, and (10)–(12) follow immediately from (13). □

Proposition 5 clearly follows from

Lemma 3. *Let* $\sigma = 1/(2\tau^2)$. *There exists* $i_0 \in \mathbb{N}$, *such that if* $i \geq i_0$ *and* $M_{n_i-1} \leq 1/q_{n_i}^{1-\sigma}$, *then we have*

$$\tilde{M}_{n_{i+1}-1} \leq \frac{1}{\tilde{q}_{n_{i+1}}^{1-\sigma}}, \quad (14)$$

and

$$\tilde{U}_{n_{i+1}-1} \leq a_i U_{n_i-1} \quad (15)$$

with $a_i \geq 1$, *and* $\Pi_{i\geq i_0} a_i < \infty$.

Proof of Lemma 3. From (10) we have

$$\begin{aligned}\tilde{M}_{l_{i+1}-1} &\leq (1+\frac{\beta_{l_{i+1}-1}}{\theta_{n_i-1}})M_{n_i-1}\\ &\leq (1+2\frac{q_{n_i}}{\tilde{q}_{l_{i+1}}})\frac{1}{q_{n_i}{}^{1-\sigma}}\end{aligned}$$

hence (6) implies for i sufficiently large

$$\tilde{M}_{l_{i+1}-1} \leq \frac{3}{\tilde{q}_{l_{i+1}}^{1/2}}. \quad (16)$$

Now if we let $K = 2[\tau]+2$ in Proposition 4, then if $i \geq i_0$, i_0 sufficently large, we obtain from (8), (7) and (5) that

$$\tilde{M}_{l_{i+1}} \leq \tilde{M}_{l_{i+1}-1}\frac{\beta_{l_{i+1}}}{\beta_{l_{i+1}-1}}(1+\tilde{q}_{l_{i+1}}^{-1/5})$$

and by induction

$$\tilde{M}_{n_{i+1}-1} \leq b_i \tilde{M}_{l_{i+1}-1}\frac{\beta_{n_{i+1}-1}}{\beta_{l_{i+1}-1}} \quad (17)$$

with $b_i \geq 1$ and $\Pi_{i\geq i_0} b_i < \infty$. Thus, (14) follows from (6).

By the same token, from (9) in Proposition 4 and (16) we get for $i \geq i_0$, i_0 sufficently large

$$\tilde{m}_{n_{i+1}-1} \geq c_i \tilde{m}_{l_{i+1}-1}\frac{\beta_{n_{i+1}-1}}{\beta_{l_{i+1}-1}}$$

with $c_i \le 1$ and $\Pi_{i\ge i_0} c_i > 0$. Together with (17) this implies that

$$\tilde{U}_{n_{i+1}-1} \le d_i \tilde{U}_{l_{i+1}-1}$$

with $d_i \ge 1$ and $\Pi d_i < \infty$. This, with (12) and (6), imply (15). □

6.3. Moving towards the "local" situation. Proof of C^1-conjugation in a special case with long Diophantine strings. The main ingredient in improving the bound of M_i towardss the "local" condition of Proposition 5 is the following.

Let $A_i \ge \tau^4$ and $B_i \ge \tau^4$ be such that

$$q_{n_{2i}} = q_{l_{2i}}^{A_i}, \quad , \tilde{q}_{n_{2i+1}} = \tilde{q}_{l_{2i+1}}^{B_i}.$$

Lemma 4. *For any $b \in \mathbb{N}$, there exists i_0 such that if $i \ge i_0$ and $u_i > 0$ is such that $M_{l_{2i}-1} = 1/q_{l_{2i}}^{u_i}$, then we have*

$$M_{n_{2i}-1} \le 1/q_{n_{2i}}^{\rho_i}$$

with $\rho_i = \min(1-\sigma, A_i^b u_i)$.

An immediate consequence of Proposition 5 and Lemma 4 is the C^1-conjugacy in the particular case of very long Diophantine strings, namely if there exist $\epsilon > 0$ and a strictly increasing subsequence of the even integers $(i_j)_{\{j\in\mathbb{N}\}}$, such that

$$q_{n_{i_j}} \ge q_{l_{i_j}}^{(\|nq_{l_{i_j}})^{\epsilon}}.$$

Proof of Lemma 4. We denote $l = l_i$ and $n = n_i$. Let r be such that

$$\frac{A_i}{\tau^4} \le \tau^{4r} \le A_i.$$

Let $\tilde{K} := 2[\tau^{4b+1}]$, so that $\tilde{K}^r \ge A_i^b$. In Proposition 4 take $K := [4\tau\tilde{K}]$.

Notice that $q_l^{(\tau^{4r})} \le q_l^{A_i} \le q_n$. Hence, we can introduce a sequence of integers p_s, $s = 0, \ldots, r$, such that $p_0 = l$, and for each $1 \le s \le r$

$$q_{p_{s-1}}^{\tau^3} \le q_{p_s} \le q_{p_{s-1}}^{(\tau^4)}.$$

Using the first estimate of Proposition 4, and following the idea of [6, Sec. 7.4] it is easy to construct for $j \in [l, n]$, positive sequences u_j and $a_j \le 2$ such that $u_l = 1/\|nq_l$, $a_l = 1$, and for $j \in [l-1, n-1]$, $M_j \le a_{j+1}/q_{j+1}^{u_{j+1}}$, where for each $j \in [l, n-1]$ one of the two following alternatives holds:

(i) If $\theta_j/\theta_{j-1} \le CM_{j-1}^{K/2}$ then $a_{j+1} = a_j$ and $u_{j+1} = \tilde{K}u_j$;

(ii) If $\theta_j/\theta_{j-1} > CM_{j-1}^{K/2}$ then $a_{j+1} = b_j a_j$ and $u_{j+1} = u_j$, with $\Pi b_j \le 2$. In this case, we actually have $M_j \le b_j M_{j-1}\theta_j/\theta_{j-1}$.

Now, if there exists $s \in [0, r-1]$ such that for every $j \in [p_s, p_{s+1}-1]$, alternative (ii) holds, then (assuming without loss of generality that $\tau \ge 2$) we have

$$\begin{aligned} M_{p_{s+1}-1} &\le 2M_{p_s-1}\frac{\theta_{p_{s+1}-1}}{\theta_{p_s-1}} \\ &\le \frac{q_{p_s}}{q_{p_{s+1}}} \\ &\le \frac{1}{q_{p_{s+1}}^{1-\sigma}} \end{aligned}$$

after which, and as in the proof of Lemma 3, only alternative (ii) can happen for all $j \in [p_{s+1}-1, n-1]$, so that, arguing again as in Lemma 3, we get $M_{n-1} \leq \frac{1}{q_n^{1-\sigma}}$ and we finish.

Otherwise, we have for every $s \in [0, r-1]$, at least one $j \in [p_s, p_{s+1}-1]$ for which alternative (i) holds, hence $u_{p_{s+1}} \geq \tilde{K} u_{p_s}$. Subsequently, $u_{p_r} \geq \tilde{K}^r u_l \geq A_i^b u_l$. The Lemma is thus proved. $\square$

6.4. Proof of C^1-conjugation in the general case. Recall that $A_i \geq \tau^4$ and $B_i \geq \tau^4$ are such that

$$q_{n_{2i}} = q_{l_{2i}}^{A_i}, \quad , \tilde{q}_{n_{2i+1}} = \tilde{q}_{l_{2i+1}}^{B_i}.$$

Then, clearly at least one of the following two limits holds

$$\limsup \frac{\Pi_{j=1}^{i} A_j^2}{\Pi_{j=1}^{i} B_j} = +\infty \tag{18}$$

$$\limsup \frac{\Pi_{j=1}^{i} B_j^2}{\Pi_{j=1}^{i} A_j} = +\infty \tag{19}$$

We will assume that (18) holds, the other case being similar. We will show how Lemma 4 applied with $b = 2$, implies that eventually the condition of Proposition 5 will be satisfied, thus yielding C^1-conjugacy.

Notice first that $q_{l_{2(i+1)}} \leq q_{n_{2i}}^{B_i}$. Furthermore, $M_{l_{2(i+1)}-1} \leq M_{n_{2i}-1}$ since $q_{l_{2(i+1)}} \geq q_{n_{2i}}^{\tau}$.

Now, if i_0 is some sufficienlty large ineteger, and if at step i_0 we do not have $M_{n_{2i_0}-1} \leq 1/q_{n_{2i_0}}^{1-\sigma}$, we observe as above that $M_{l_{2(i_0+1)}-1} \leq M_{n_{2i_0}-1} \leq 1/q_{l_{2(i_0+1)}}^{u_{i_0} A_{i_0}^2 / B_{i_0}}$. A continued application of the Lemma hence shows that either at some $i \geq i_0 + 1$ the condition of Proposition 5 will be satisfied, or for every $i \geq i_0$, $M_{l_{2i}-1} \leq 1/q_{l_{2i}}^{u_{i_0} \Pi_{j=i_0}^{i-1} (A_j^2/B_j)}$ which, with our assumption that (18) holds, contradict the fact that for every i, $M_{l_{2i}-1} \geq 1/(2q_{l_{2i}})$.

Remark 3. In the general situation, the alternated configuration of denominators may require the use of more than two angles, that is more than two diffeomorphisms. Our proof remains quite the same. Indeed, let $\theta_1, \dots, \theta_p$ be in an alternated configuration as in definition 2. Define A_i such that $q_{j_i, n_i} = q_{j_i, l_i}^{A_i}$. Then there exists $k \in [1, p]$ such that

$$\limsup_{I \in \mathbb{N}} \frac{\Pi_{j_i = k, i \leq I} A_i^{p+1}}{\Pi_{j_i \neq k, i \leq I} A_i} = +\infty$$

and the proof of C^1-conjugation follows the same lines as above with this difference that we would take $b = p+1$ in Lemma 4, $\tilde{K} = 2[\tau^{4b+1}])$ and then $K := [4\tau\tilde{K}]$ in Proposition 4 which is possible since the diffeomorphisms we are considering are of class C^∞. We see here the dramatic increase in our need of differentiability to prove C^1-conjugation as the number d of commuting diffeomorphisms in Theorem 1 increases.

7. Higher regularity

We fix $r \geq 2$. Knowing that the diffeomorphisms f and g are C^1-conjugated to the rotations, we will now prove that the conjugacy is in fact of class C^r.

In all the sequel, we fix $k = [(r+2)(2+\tau)] + 2$. And we take $\xi = 1/k$ in Proposition 2.

As in the proof of C^1-conjugation, we will continue to assume for simplicity that we are given θ and β satisfying (4)–(6) with in addition that there exists for each i, l'_i such that if i is even, then

$$q_{l'_i} \leq q_{l_i}^{1/k}, \quad \text{and } l'_i, \dots, l_i - 1 \in \mathcal{A}_\tau(\theta), \tag{20}$$

with a similar property involving β if i is odd.

Given two commuting diffeomorphisms $f \in \mathcal{D}_\theta$, $g \in \mathcal{D}_\beta$ such that f and g are C^1-conjugated, we will show that the conjugacy is actually of class C^r.

7.1. The control of the derivatives at alternating "Diophantine times" is sufficient. We define two sets of integers, the "Diophantine times", as

$$\begin{aligned}
\mathcal{A} &= \{m \in \mathbb{N} \,/\, m = \sum a q_s \text{ with } s \in [l_{2i}, n_{2i} - 1],\ i \in \mathbb{N},\ a \leq q_{s+1}/q_s\} \\
\tilde{\mathcal{A}} &= \{m \in \mathbb{N} \,/\, m = \sum a \tilde{q}_s \text{ with } s \in [l_{2i+1}, n_{2i+1} - 1],\ i \in \mathbb{N},\ a \leq \tilde{q}_{s+1}/\tilde{q}_s\}.
\end{aligned}$$

We also define two sets of diffeomorphisms

$$\begin{aligned}
\mathcal{Z} &= \{f^n \,/\, n \in \mathbb{N}\} \\
\mathcal{C} &= \{f^u \circ g^v \,/\, u \in \mathcal{A}, v \in \tilde{\mathcal{A}}\}.
\end{aligned}$$

The following is an elementary Lemma due to (6)

Lemma 5. *If we denote*

$$\mathcal{O} = \{u\theta + v\beta \mathrm{mod}[1] \,/\, u \in \mathcal{A}, v \in \tilde{\mathcal{A}}\}$$

then $\overline{\mathcal{O}} = \mathbb{T}$.

As a consequence, we have that $\mathcal{C}$ is dense in $\mathcal{Z}$ in the C^0-topology.

It follows from the above Lemma that it is enough to control the derivatives of the f^u and g^v at the *Diophantine times* $u \in \mathcal{A}$ and $v \in \tilde{\mathcal{A}}$:

Corollary 3. *If $\mathcal{C}$ is bounded in the C^{r+1}-topology, then the conjugating diffeomorphism h of f to R_θ is of class C^r.*

Proof. We know that $\frac{1}{n}\sum_{i=0}^{n-1} f^i$ converges in the C^0-topology to h (see [4, chap. IV]). From Lemma 5, this implies that there exist sequences (u_n) and (v_n) of numbers in $\mathcal{A}$ and $\tilde{\mathcal{A}}$ such that the sequence $\frac{1}{n}\sum_{i=0}^{n-1} f^{u_i} \circ g^{v_i}$ converges in the C^0 topology to h. By our C^{r+1}-boundness assumption, we can extract from the latter sequence a sequence that converges in the C^r-topology, so that necessarily $h \in \mathrm{Diff}^r_+(\mathbb{T})$. □

7.2. It follows from standard computations (see [6, section 8.10]) that the assumption of corollary 3 holds true if we prove

Lemma 6. *There exists $\nu > 0$ such that, for i (even) sufficenlty large, we have for any $s \in [l_i, n_i - 1]$ and for any $0 \leq a \leq q_{s+1}/q_s$*

$$\| \ln D f^{a q_s} \|_{r+1} \leq q_s^{-\nu}$$

(with g and $\tilde{q}_s$ instead of f and q_s if i is odd).

Proof. We will only work with f since the arguments for g are the same. The proof is based on the estimates of [6, section 8] and we start by recalling some facts that were proven there:

For $k \in \mathbb{N}^*$, define for $s \in \mathbb{N}$, $\Delta_s^{(k)} = \|D^{k-1} \ln Df^{q_s}\|_0 + \theta_s$. Then it follows from the C^1-conjugation of f to R_θ (see [6, lemme 5]) that

$$\Delta_s^{(k)} \leq q_s^{(k-1)/2}.$$

We will use this fact with $k = [(r+2)(2+\tau)] + 2$ and use the notation Δ_s for $\Delta_s^{(k)}$.

Observe that for $s \in [l'_i, n_i - 1]$, we have if i is sufficiently large

$$(\Delta_s q_{s+1})^{1/k} q_s^{-1} \leq q_s^{-1/4}. \tag{21}$$

Hence it follows from [6, lemme 14 in section 8.8] that for any $s \in [l'_i, n_i - 1]$, and for any $0 \leq a \leq q_{s+1}/q_s$, we have

$$\| \ln Df^{aq_s}\|_{r+1} \leq C q_s^{-1} (\Delta_s q_{s+1})^\rho \tag{22}$$

where $\rho = (r+2)/k$ and C is some constant.

If we denote

$$\Delta'_s = \mathrm{Sup}\{|(D^{k-1} \|n Df^{q_t} \circ f^m)(Df^m)^{k-1}|_0,\ 0 \leq t \leq s, m \geq 0\},$$

then we have $\Delta_s \leq C\Delta'_s$ for some constant C. Observe that since $\|Df^m\|_0$ is bounded we have $\Delta'_s \leq C q_s^{(k-1)/2}$ for some constant C. If we denote $V_s = \mathrm{Max}\{\Delta'_t/q_t,\ 0 \leq t \leq s\}$, then due to (21) we have from [6, section 8.9] that for $s \in [l'_i, n_i - 1]$,

$$V_{s+1} \leq V_s(1 + C q_s^{-1/4})$$

for some constant C. Hence, for $s \in [l'_i, n_i - 1]$, we have $V_s \leq 2V_{l'_i} \leq C q_{l'_i}^{(k-3)/2}$. If $s \geq l_i$ this gives

$$\begin{aligned} \Delta'_s &\leq C q_s q_{l'_i}^{(k-3)/2} \\ &\leq C q_s^2 \end{aligned}$$

because we assumed that $q_{l'_i} \leq q_{l_i}^{2/k}$.

Finally, if $s \in [l_i, n_i - 1]$ we have that $(\Delta_s q_{s+1})^\rho \leq C q_s^{(2+\tau)(r+2)/k} \leq q_s^{1-1/k}$ and we conclude using (22) that the statement of Lemma 6 holds which ends the proof of higher regularity. □

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E-mail address: fayadb@math.univ-paris13.fr
E-mail address: khanin@math.toronto.edu